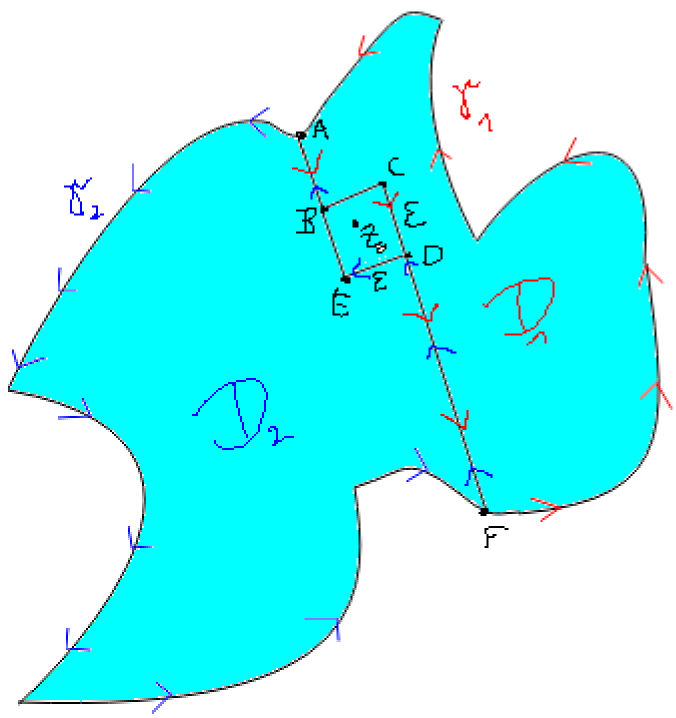
3 Cauchy's integral theorem II and Cauchy's integral formula

In this chapter, we extend Cauchys theorem from the last chapter a little bit and derive a corollary from it (another theorem of Cauchy).

In what follows, we let $f : U \subset \mathbb{C} \to \mathbb{C}$ with U open be a continuous function. Within U we assume lies a piecewise C^1 curve γ of Jordan type (in the picture below, this is represented as the boundary of the green colored region) together with its interior $\text{int}(\gamma)$. We will further assume f to be holomorphic on U except for one point z_0 in the interior of γ.

Since z_0 is an interior point, we may draw a small (closed) square $BCDE$ (see the following picture) of side length $\epsilon > 0$ around z_0 which lies in the interior of γ. Extending the lines EB and CD to the boundary, we get two piecewise C^1 curves of Jordan type, namely $\gamma_1 = \gamma_{1,\epsilon}$, following the red arrows and traveling along A, B, C, D, F, A, and $\gamma_2 = \gamma_{2,\epsilon}$, following the blue arrows through A, F, D, E, B, A, with interiors D_1 and D_2:



Now we let $\gamma_0 = \gamma_{0,\epsilon}$ be the closed curve we get when we travel along B, C, D, E, B. With the help of Theorem 2.1 from the previous chapter applied to f and the curves $\gamma_{1,\epsilon}$ and $\gamma_{2,\epsilon}$ respectively, it follows that

$$0 = \int_{\gamma_{1,\epsilon}} f dz + \int_{\gamma_{2,\epsilon}} f dz = \int_{\gamma_{0,\epsilon}} f dz + \int_{\gamma} f dz,$$

where we use that in the first sum the integrals over the lines AB and DF cancel each other out since we integrate in different directions. So, since f is continuous and therefore bounded on the closure of $\mathrm{int}(\gamma)$ (which is compact), letting $c := \sup_{z \in \mathrm{int}(\gamma)} |f(z)| < \infty$ gives

$$\left| \int_{\gamma_{0,\epsilon}} f dz \right| \leq \mathrm{length}(\gamma_{0,\epsilon}) \cdot c = 4\epsilon c \to 0$$

as $\epsilon \searrow 0$. So we have the following extension of our theorem of Cauchy we presented in the last chapter (which is also due to Cauchy):

Theorem 3.1. *(Cauchy's integral theorem) We let γ be a positively oriented piecewise C^1 curve of Jordan type. We let further $f : U \subset \mathbb{C} \to \mathbb{C}$ be a continuous function where U is open and contains the closure of the interior $int(\gamma)$ of γ. Then if f is holomorphic on U except for possibly one point z_0 in $int(\gamma)$, we have*

$$\int_{\gamma} f(z) dz = 0.$$

Denoting $D(r, z_0) \subset \mathbb{C}$ the (open) disc of radius r and center z_0 and $\partial D(r, z_0)$ its positively oriented boundary, we may easily deduce:

Theorem 3.2. *(Cauchy's integral formula) Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Let z_0 be in U and $r > 0$ for which the closure of $D := D(r, z_0)$ lies in U, then we have for every z in D*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} dw.$$

Proof. Let z be in D and let

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{for } w \neq z \\ f'(w) & \text{for } w = z \end{cases} .$$

Since f is holomorphic on U, g is both holomorphic on $U - \{z\}$ and continuous on $w = z$ and therefore on U. Applying g and ∂D to Theorem 3.1 above and recalling that in the previous chapter we found that $\int_{\partial D} 1/(w - z) dw = 2\pi i$, we get the assertion of our theorem:

$$\int_{\partial D} \frac{f(w)}{w - z} dw = \int_{\partial D} g(w) dw + \int_{\partial D} \frac{f(z)}{w - z} dw = 2\pi i f(z).$$

□

Note that Theorem 2 is rather remarkable. By differentiating under the integral sign with respect to z (we leave it to the reader to check that we are allowed to do this) and therefore essentially differentiating the function $1/(w-z)$ with respect to z, we find that *a holomorphic function is in fact infinitely many often complex differentiable* (!):

Corollary 3.3. *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function, then f is infinitely often complex differentiable with*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{n+1}} dw,$$

where $D = D(r, z_0)$ is as in Theorem 2.