4 Parameter integrals, power series and analytic continuation

In this lecture, we first discuss two methods of how we can get new holomorphic functions out of old ones. The first method is via a parameter integral:

Proposition 4.1. *Suppose that we have a piecewise C^1 curve $\gamma : [a, b] \to \mathbb{C}$ and a continuous function $F : im(\gamma) \times U \to \mathbb{C}$ with $U \subset \mathbb{C}$ open. Suppose further that for every $w_0 \in im(\gamma)$ the function $z \mapsto F(w_0, z)$ on U is complex differentiable and that the function $(w, z) \mapsto \partial_z F(w, z)$ is continuous. Then the function*

$$G : U \to \mathbb{C}, z \mapsto \int_\gamma F(w, z)dw$$

is also complex differentiable with $G'(z) = \int_\gamma \partial_z F(w, z)dw$.

Proof. Let z be in U. Since F is continuous, we have that for every disc $D = D(0, r)$ with center zero for which $D + z \subset U$ is relatively compact, the function

$$\psi(w, h) := \begin{cases} F(w, z + h) - F(w, z)/h & \text{for } h \neq 0 \\ \partial_z F(w, z) & \text{for } h = 0 \end{cases}$$

is uniformly continuous on $\mathrm{im}(\gamma) \times D$. So integrating $\psi(w, h)$ along γ and letting $h \to 0$, we get the assertion of our proposition. □

As a second method we now show that we may get holomorphic functions out of locally uniformly convergent series of holomorphic functions:

Proposition 4.2. *Suppose we have a sequence $(f_n)_{n \geq 0}$ of holomorphic functions $U \subset \mathbb{C} \to \mathbb{C}$. Then, if the infinite series $\sum_{n=0}^{\infty} f_n$ converges locally uniformly on U to the function $f : U \to \mathbb{C}$, then f itself is holomorphic and we have that $f^{(n)}$ is given by the infinite series $\sum_{k=0}^{\infty} f_k^{(n)}$, which is also locally uniformly convergent.*

Proof. First note that since $f = \sum f_k$ is locally uniformly convergent and the f_k are in particular continuous, f must be continuous (proof is left to the reader). Let $D \subset U$ be a relatively compact disc of radius > 0 with positively oriented boundary ∂D, then (f_k) converges on D uniformly. By definition this means that given $\epsilon > 0$, there is an integer N such that

$$|f(z) - \sum_{k \leq m} f_k(z)| < \epsilon$$

for all $m \geq N$ and $z \in D$. Choosing $n \geq 0$ and letting

$$F_n(z) := \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{n+1}} dw,$$

which is of course holomorphic on D by Proposition 4.1, then for every compact subset K of D and every z in K

$$\left| F_n(z) - \sum_{k \leq m} f^{(n)}(z) \right| \leq \frac{n!}{2\pi} \int_{\partial D} \left| \frac{f(w) - \sum_{k \leq m} f_k(w)}{(w-z)^{n+1}} \right| dw \leq C_n \cdot \epsilon,$$

where $C_n = n!\text{length}(\partial D)/2\pi\text{dist}(K, \partial D)^{n+1} > 0$ with

$$\text{dist}(K, \partial D) := \inf\{|z_1 - z_2| : z_1 \in K, z_2 \in \partial D\} > 0$$

and where we have used Cauchy's integral formula for the $n-$th derivative of a holomorphic function. Now, since D, ϵ and K were chosen arbirary, we see for all n that $F_n = \sum f_k^{(n)}$ locally uniformly, but since $F_0 = f$ by assumption, we have that f is holomorphic and that $F_n = f^{(n)}$ for all $n \geq 0$. □

We will now look at complex power series. Given a point z_0 in the complex plane, then a *complex power series in z around z_0* is an infinite series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the numbers a_n are in $\mathbb{C}$. Recall that for a sequence $(b_n)_{n \geq 0}$ of real numbers one defines the *limes superior*, written $\overline{\lim}_{n \geq 0} b_n$, to be

$$\lim_{n \to \infty} \sup_k \{b_k : k \geq n\}$$

which, being the limes of a monotonically decreasing sequence, is a unique element in $[-\infty, \infty]$. To f above we may associate a unique element R out of $[0, \infty]$ given by $R = 1/L$, where $L = \overline{\lim}_{n \geq 0} \sqrt[n]{|a_n|}$. R is called the *convergence radius* of f. Furthermore, the open disc $D = D(z_0, R)$ around z_0 of radius R will be called the *(open) disc of convergence* of f. The names we gave R and D are justified by the following well known result on power series:

Theorem 4.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a complex power series with convergence radius R. Let further $D = D(z_0, R)$ be the disc of convergence of f. Then f converges absolutely and locally uniformly on D. It may converge at points on the boundary ∂D of D and it diverges outside the closure $\overline{D}$ of D.*

Proof. (omitted) □

Let us give two more facts about complex power series which we will use below:

Proposition 4.4. *Let $f(z) = \sum a_n(z - z_0)^n$ be a complex power series. Then the following hold:*

1. *f restricted to its disc of convergence gives a holomorphic function.*
2. *Let $g(z) = \sum b_n(z - z_0)^n$ be a second complex power series, and assume that both f and g have a positive radius of convergence. Suppose further that for a sequence $(w_k)_{k \geq 0}$ of points in $\mathbb{C} - \{z_0\}$ which converges to z_0 we have $f(w_k) = g(w_k)$ for almost all $k \geq 0$, then $a_n = b_n$ for all $n \geq 0$.*

Proof. Assertion 1. follows directly from Proposition 4.2 where we let $f_n(z) = a_n(z - z_0)^n$. For assertion 2., note first that $\lim f(w_k) = f(z_0) = g(z_0) = \lim g(w_k)$, i.e. we have $a_0 = b_0$. Assuming now that $a_k = b_k$ for $k = 0, 1, 2, ..., n$ for some $n \geq 0$, then the function

$$F(z) := \frac{f(z) - g(z)}{(z - z_0)^{n+1}}$$

is locally continuous around z_0 and we have $F(w_k) = 0$ for almost all k. So we have $\lim F(w_k) = 0$, i.e. $a_{n+1} = b_{n+1}$. By induction 2. follows. $\square$

Having introduced complex power series, we may now formulate another characterization of a function being holomorphic.

Theorem 4.5. *A function $f : U \subset \mathbb{C} \to \mathbb{C}$ is holomorphic iff at every point z_0 in U and for every $R > 0$ such that the closure of $D(z_0, R)$ lies in U, we have that f restricted to $D(z_0, R)$ is a complex power series around z_0 with a convergence radius larger than R.*

Proof. "$\Rightarrow$": Suppose f is holomorphic, then by Corollary 3.3 of the previous chapter, f is infinitely many times complex differentiable. So given z_0 in U, we may consider the (formal) power series

$$F(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Using the integral representation of $f^{(n)}(z_0)/n!$ from Corollary 6.3, then for every $D = D(z_0, R)$ whose closure lies in U, we have

$$\left|\frac{f^{(n)}(z_0)}{n!}\right| = \left|\frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{(z - z_0)^{n+1}} dz\right| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot C \cdot R^{-(n+1)} = C/R^n.$$

where $C = \sup_{z \in D} |f(z)|$. From this we get $\overline{\lim}_{n \geq 0} \sqrt[n]{\left|\frac{f^{(n)}(z_0)}{n!}\right|} \leq 1/R$, using that for a positive constant c we have $\sqrt[n]{c} \to 1$ as $n \to \infty$. Since we may make D a little larger by slightly increasing R and have that the closure still stays in U, we see that f restricted to D is a complex power series around z_0 with convergence radius greater than R.

"$\Leftarrow$": Since at every point z_0 the function f a complex power series with positive convergence radius, it is holomorphic there. So f itself is holomorphic. $\square$

Let us now end this chapter with a remarkable fact about holomorphic functions. We say that an open subset G of $\mathbb{C}$ is a *region* (*"Gebiet"* in german) iff it cannot be written as a union of two non-empty and disjoint open sets. Then the following holds:

Theorem 4.6. *(Analytic continuation) Given two holomorphic functions $f, g : G \subset \mathbb{C} \to \mathbb{C}$, where G is a region in $\mathbb{C}$. Suppose that $f(z) = g(z)$ on a sequence of distinct points with limit point in G. Then $f(z) = g(z)$ throughout G.*

Proof. Combining Theorem 4.5 and Proposition 4.4 (2.), we see that f and g agree on a small open disc $D \subset G$ of positive radius.
We let $U_1 := D$ and for $n \geq 1$ build the set U_{n+1} out of U_n as the union of all relatively compact open discs in G whose centers are in U_n. Then obviously for all $n \geq 1$ we have $U_n \subset U_{n+1} \subset G$, U_n is open and we have $f(z) = g(z)$ on U_n by using Theorem 4.5 and Proposition 4.4 (2.). We let V be the open subset $\cup_{n \geq 1} U_n$ of G and let $V' := G - V$. Suppose now there was a z_0 in V' which was a boundary point of V. Then we would have that very close to z_0 there was a point w_0 in one of the sets U_n for which a relatively compact disc in G with center w_0 existed which contained z_0 , i.e. then $z_0 \in U_{n+1} \subset V$, a contradiction. So V' contains no boundary point of V and must therefore be open since G is open. But since $G = V \cup V'$ is a region and $V \neq \emptyset$, we must have $V' = \emptyset$, which concludes our proof. $\square$