7 Meromorphic functions and products

In this lecture, we look at some well known facts about products. Given an open subset U of $\mathbb{C}$, then a *meromorphic function* on U is any holomorphic function $f : U - N_f \to \mathbb{C}$, where N_f is a discrete subset of U where f has poles. F.e., given a non-zero polynomial $p(z)$, then $f(z) = 1/p(z)$ is a meromorphic function on $\mathbb{C}$ where N_f is the set of zeros of $p(z)$. N_f in this case is not only discrete but also finite. In fact Gauss proved, as we will do also shortly using Complex analysis, that $p(z)$ is a product of (finitely many) linear factors:

Theorem 7.1. *(Fundamental Theorem of Algebra) Every complex polynomial of positive degree has a zero.*

Since for every non-zero polynomial of $p(z)$ positive degree we have $p(z) \to \infty$ as $z \to \infty$, the Theorem 7.1 follows from

Theorem 7.2. *(Liouville) Given a holomorphic function f on the complex plane which has no zeros and such that $1/f$ is bounded, then f must be constant.*

Proof. By assumption, there is a real number $c > 0$ such that $|1/f(z)| \leq c$ for all complex numbers z. Given a complex number z_0, then we have for every circle C_r of radius r and center z_0 that

$$|(1/f)'(z_0)| = \left| \frac{1}{2\pi i} \int_{C_r} \frac{1}{f(z)(z - z_0)^2} dz \right| \leq \frac{1}{2\pi} 2\pi r c r^{-2} = \frac{c}{r}$$

for all $r > 0$. Letting $r \to \infty$, we see that $(1/f)'(z_0) = 0$. And since c was arbitrary, we see that $(1/f)'(z) = 0$ for all z in the plane, i.e. $1/f$ and therefore also f must be constant. □

Obviously, given two meromorphic functions f_1, f_2 on an open subset U of the complex plane, then also their sum $f_1 + f_2$ and their (pointweise) product $f_1 f_2$ is meromorphic. Moreover, if f_1, f_2 are meromorphic with f_2 being non-trivial, i.e. not completely zero, in any region within U, then f_1/f_2 is meromorphic since by analytic continuation f_2 can only have a discrete set of zeros.

A *divisor* D on the complex plane will be any function $D : \mathbb{C} \to \mathbb{Z}$ where the support of D is a discrete subset of $\mathbb{C}$. It is sometimes convenient to write

$$D = \{(a_0, n_0), (a_1, n_1), ...\}$$

where $\{a_0, a_1, ...\}$ is a discrete set of points in $\mathbb{C}$ and where the numbers $n_0, n_1, ...$ are non-zero integers. Obviously, given a nowhere trivial meromorphic function $f(z)$ on the complex plane, we may form the divisor D_f where $D_f(a) = n$ when a is a zero of order n and $D_f(a) = -n$ if a is a pole of order n.

Question: Given an arbitrary divisor D on $\mathbb{C}$, is there a meromorphic function f with $D = D_f$?

As we will see below, the well-known answer to this question is *yes*! Since obviously

$$D_{fg} = D_f + D_g \text{ and especially } D_{1/f} = -D_f,$$

we are reduced to the case where $D(z) \geq 0$ for all z, i.e. the case where D is *positive*. Also, if $\operatorname{supp}(D)$ is a finite set, we have $D = D_{p/q}$ where $p(z)$ and $q(z)$ are appropriately chosen polynomials. So in sum we need to deal with the case where $\operatorname{supp}(D)$ is infinite (and therefore unbounded) and where D is positive.

In what follows we let $\log(z)$ be the *natural branch of the logarithm*, i.e. we let

$$\log(z) = \int_1^z \frac{1}{w} dw$$

where z is in $\mathbb{C} - \mathbb{R}_{\geq 0}$. Before we proceed let us list some properties of $\log(z)$:

Lemma 7.3. *The following are true:*

1. *We have the power series expansion around zero*

$$-\log(1-z) = \sum_{n\geq 1} \frac{z^n}{n}$$

   with radius of convergence $R = 1$.

2. *For $|z| \leq 1/4$ we have*

$$\frac{2}{3}|u| \leq |\log(1+u)| \leq \frac{4}{3}|u|.$$

3. *Let $e_N(z) = \log(1-z) + \sum_{n=1}^N \frac{z^n}{n}$. Then for every $N \geq 0$ and every $|z| \leq 1/4$ we have*

$$|e_N(z)| \leq 4/3\, |z|^{N+1}.$$

4. *Let $E_n(z) = \exp(e_N(z)) = (1-z)\exp(\sum_{k=1}^n \frac{z^k}{k})$. Then whenever $n \geq 0$ and $|z| \leq 1/4$, we have*

$$|1 - E_n(z)| \leq 2|z|^{n+1}.$$

Proof. To prove 1., note that the power series on the right clearly is absolutely and locally uniformly convergent on the open unit disc and it diverges outside its closure, so it has a convergence radius of one. Furthermore, within a small disc around zero, both sides have the same derivative, namely

$$\frac{1}{1-z} = \sum z^n.$$

In particular up to the constant term, both sides have the same power series expansion around zero. But since $\log(1-0) = 0$, they have the same.

To prove 2., we use 1.: Since for $|z| \leq 1/4$

$$\sum_{n\geq 2} \frac{|z|^n}{n} \leq \sum_{n\geq 0} |z|^{n+2} \leq 1/4 \cdot |z| \cdot \frac{1}{1-|z|} \leq 1/3|z|,$$

we have

$$2/3|z| = |z| - 1/3|z| \leq |\log(1-z)| \leq |z| + 1/3|z| = 4/3|z|.$$

Again using 1., we prove 3.:
We have

$$\left|\log(1-z) + \sum_{n=1}^{N} \frac{z^n}{n}\right| = \left|\sum_{n\geq N+1} \frac{z^n}{n}\right| \leq |z|^{N+1}\frac{1}{1-|z|} \leq 4/3|z|^{N+1}.$$

Finally, 4. is simply a consequence of 3.: We have

$$|1-E_n(z)| \leq \sum_{k\geq 1} \frac{|e_n(z)|^k}{k!} \leq \frac{4}{3}|z|^{n+1}\sum_{k\geq 0}\left(\frac{4}{3}|z|^{n+1}\right)^k = \frac{\frac{4}{3}|z|^{n+1}}{1-\frac{4}{3}|z|^{n+1}} \leq 2|z|^{n+1}.$$

□

Inspired by Lemma 7.3 (2.) and the continuity of the exponential function, we define:

Definition 7.4. We say that the infinite product $\prod_{n\geq 0}(1-a_n)$ *converges absolutely* iff $\sum \log(1-a_n)$ converges absolutely. Moreover, we say that a product $\prod(1-f_n(z))$ converges *absolutely and locally uniformly* on a subset U of the complex plane iff $\sum \log(1-f_n(z))$ does.

Clearly, in the case of convergence, the product $\prod(1-f_n(z))$ will be a holomorphic function whenever the functions $f_n(z)$ are. Let us now come to the main theorem of this chapter:

Theorem 7.5. *(Weierstrass) Given a divisor*

$$D = \{(a_0,n_0),(a_1,n_1),(a_2,n_2),...\},$$

where $0 = |a_0| < |a_1| \leq |a_2| \leq |a_3| \leq ...$ *and* $a_n \to \infty$ *as* $n \to \infty$*, then the function*

$$f(z) := z^{n_0}\prod_{k\geq 1}(E_{k+n_k}(z/a_k))^{n_k}$$

defines a holomorphic function on the complex plane and satisfies $D = D_f$*. If* $g(z)$ *is another such function then* $f/g = e^h$ *for a holomorphic function* h *on* $\mathbb{C}$*.*

Proof. Let $R > 0$. Since $a_k \to \infty$ as $k \to \infty$, there is a constant $k_0 \geq 0$ such that $|z/a_k| \leq 1/4$ for all $k \geq k_0$ and all z with $|z| \leq R$. So using Lemma 7.3 (4.), we have

$$\begin{array}{rcl}\sum_{k\geq k_0} n_k\,|E_{k+n_k}(z/a_k)-1| & \leq & \sum_{k\geq k_0} 2n_k\,|z/a_k|^{k+n_k+1} \\ & \leq & \sum_{k\geq k_0}\frac{n_k}{4^{n_k}}\cdot\frac{1}{4^{k+1}} \\ & \leq & \frac{1}{1-1/4} = 4/3 < \infty.\end{array}$$

This means that $f(z)$ converges absolutely and uniformly on the closed disc around zero with radius R, and since R was arbitrary $f(z)$ forms an absolutely and locally uniformly convergent product of holomorphic functions and is therefore itself holomorphic. Obviously, by construction, we have $D = D_f$. Finally, from what we know about logarithms, if the holomorphic function $g(z)$ on the complex plane also satisfies $D = D_g$ then the meromorphic function f/g defines a function which has no poles and no zeros on $\mathbb{C}$ and therefore has a logarithm on $\mathbb{C}$. This proves the last assertion of the theorem. $\square$