8 The gamma function and the Riemann zeta function

In this chapter, we get a first glance at the famous Riemann zeta function and thereby give examples for the topics we have developed so far. Our introduction to this function is absolutely standard and follows Riemanns very beautiful article [1], where beautiful refers to both its very influential and important content and also in how it is written; this article is de facto a master piece, in it, Riemann formulates what is today known as the *Riemann hypothesis.*

Given a complex number $s = \sigma + it$ with $\sigma > 1$, we introduce the *Riemann zeta function* ζ as the infinite series

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}.$$

Since $|n^{-s}| = |n^{-\sigma}| \leq \int_{n-1}^{n} x^{-\sigma} dx$ for $n > 1$, we see that $|\zeta(s)| \leq 1 + \int_{x>1} x^{-\sigma} dx = \sigma/(\sigma - 1)$ and therefore that $\zeta(s)$ converges absolutely and locally uniformly when $\sigma > 1$ and consequently forms a holomorphic function by Proposition 4.2. Using the uniqueness of the factorization of natural numbers into prime powers, we immediately get the important fact that for $\sigma > 1$ we have

$$\zeta(s) = \sum n^{-s} = \prod_{p} \sum_{k\geq 0} p^{-ks} = \prod_{p} (1 - p^{-s})^{-1}$$

where p runs through the set of prime numbers (this f.e. has been used by Euler in the case where s is real).

To learn more about $\zeta(s)$ as a complex function, Riemann used the *Gamma function* which is defined as the integral

$$\Gamma(s) := \int_{u>0} e^{-u} u^{s-1} du.$$

This integral converges absolutely for every $\sigma > 0$ as a simple calculation shows. Then $\Gamma(s) = \sum_{n\geq 1}(\Gamma_{n+1}(s) - \Gamma_n(s))$ where $\Gamma_m(s) := \int_{1/m}^{m} e^{-u} u^{s-1} du$ converges absolutely and uniformly. Using Proposition 4.1 and Proposition 4.2, we see that $\Gamma(s)$ forms a holomorphic function for all s where $\sigma > 0$.

Now, using integration by parts, we see that

$$\Gamma(s) = e^{-u} u^{s}/s|_{0}^{\infty} + \int_{u>0} e^{-u} u^{s}/s du = \frac{1}{s}\Gamma(s+1)$$

for $\sigma > -1$ forms a meromorphic function with a simple pole at $s = 0$, since $\Gamma(1) = \int_{u>0} e^{-u} du = 1 \neq 0$. Repeatedly applying this technique, we see that we may extend $\Gamma(s)$ meromorphically to the whole complex plane with simple poles at $s = 0, -1, -2, -3, ...$ and satisfying the functional equation

$$\Gamma(s + 1) = s\Gamma(s).$$

Now let us see how cleverly Riemann shows that ζ also can be extended meromorphically to the complex plane and that it also satisfies a functional equation. We let

$$\Theta(x) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x} = 1 + 2\omega(x)$$

where

$$\omega(x) = \sum_{n \geq 1} e^{-\pi n^2 x}.$$

Both Θ and ω converge uniformly for $x \geq \epsilon > 0$. Moreover, one can check (may be we will do that in a later chapter using complex analysis) that Θ which is an example of a so called "theta function" satisfies the functional equation

$$\Theta(1/x) = \sqrt{x}\Theta(x)$$

or equivalently

$$\omega(1/x) = -1/2 + \sqrt{x}/2 + \sqrt{x}\omega(x).$$

Now using the uniform convergence of ω, we see that

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-s/2} \sum_{n \geq 1} n^{-s} \int_{u>0} e^{-u} u^{s/2-1} du = \int_{v>0} \omega(v) v^{s/2-1} dv,$$

where we get the last equality by letting $u = \pi n^2 v$. The right hand side of the last equation is equal to

$$\int_0^1 \omega(v) v^{s/2-1} dv + \int_1^\infty \omega(v) v^{s/2-1} dv$$

which after letting $v \mapsto 1/v$ in the first integral and applying the functional equation of ω and summarizing terms results in the identity

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = -1/s + 1/(s-1) + \int_1^\infty \omega(v)(v^{s/2-1} + v^{-s/2-1/2}) dv. \quad (8.1)$$

The identity (8.1) is remarkable: The integral on the right hand side converges for all s locally uniformly (note that $\omega(x) = O(e^{-\pi x})$) and therefore forms a holomorphic function on the complex plane. Since Γ has a (simple) pole at zero, we see therefore that ζ is in fact a meromorphic function on the whole complex plane with only one simple pole at $s = 1$. But not only that, the right hand side of (8.1) is invariant under the transformation $s \mapsto 1 - s$, so we get a functional equation. Now consider the zeros of ζ for $\sigma < 0$: ζ must have zeros of order one at $s = -2, -4, -6, \ldots$ since the right hand side of (8.1) is holomorphic for $\sigma < 0$, obviously positive (i.e. non-zero) for real $s < -1$ and Γ has simple poles at $s = -1, -2, \ldots$. Any other zero at $\sigma < 0$ of ζ would lead by the functional equation to a zero of Γ at $\sigma > 1/2$ or ζ at $\sigma > 1$ where as one checks they have no zeros. So we have the following result:

Theorem 8.1. *(Riemann) $\zeta(s)$ can be meromorphically extended to the whole complex plane. More precisely, in $\mathbb{C} - \{1\}$, $\zeta(s)$ is holomorphic and it has a simple pole at $s = 1$. Moreover, we have*

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s).$$

For $\sigma < 0$, $\zeta(s)$ has precisely the zeros $s = -2, -4, -6, ...$ and they are of order one.

Remark 8.2. There is of course a lot more to say, f.e. the zeros mentioned in the theorem have all order one. And there is the critical strip we have not mentioned yet. This will come hopefully later ...

[1] Riemann, Bernhard, Ueber die Anzahl der Primzahlen unter einer gegebenen Groesse, Monatsberichte der Koeniglichen Preussischen Akademie der Wissenschaften zu Berlin. Aus dem Jahre 1859. S. 671–680.

https://www.claymath.org/sites/default/files/zeta.pdf