1 Holomorphic functions

Before we start, let us very briefly recall the complex numbers. We define the *complex numbers*, denoted $\mathbb{C}$, to be the set of pairs (x, y) of real numbers together with the following two binary operations, called the "addition" and the "multiplication" of $\mathbb{C}$, defined by

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

and

$$(x_1, y_1) \cdot (x_2, y_2) := (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$$

for all real numbers x_i, y_i $(i = 1, 2)$. Using the laws of distributivity and commutativity in $\mathbb{R}$, one easily verifies that the operations $+, \cdot$ introduced above are also distributive and commutative. Furthermore, one checks that $(\mathbb{C}, +)$ and $(\mathbb{C} - \{(0, 0)\}, \cdot)$ form abelian groups with neutral elements $(0, 0)$ and $(1, 0)$ respectively. F.e. let $z = (x, y) \neq 0$, then the multiplicative inverse of (x, y) is given by

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

So $\mathbb{C}$ is in fact a field. Moreover, the map

$$\phi : \mathbb{R} \ni x \mapsto (x, 0) \in \mathbb{C}$$

is an injection of $\mathbb{R}$ into $\mathbb{C}$, i.e. it is an injective map which is "structure preserving" in the sense that

$$\phi(x_1 + x_2) = \phi(x_1) + \phi(x_2) \text{ and } \phi(x_1 x_2) = \phi(x_1)\phi(x_2)$$

for all x_1, x_2 in $\mathbb{R}$. So we may identify $\mathbb{R}$ as a subfield of $\mathbb{C}$, which we will do in what follows. Now, if we introduce the vector $i = (0, 1)$, then $\mathbb{C}$ as a real vector space of dimension two will have $1 = (1, 0)$ and i as a basis so that every z in $\mathbb{C}$ can be uniquely written as $z = x + yi$ where x, y are real numbers. Also, we introduce the *complex conjugate* $\bar{z}$ of the complex number $z = x + yi$ as the number $x - yi$. Note that $z\bar{z} = x^2 + y^2$. We let $|z| := \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$ and get the Euclidean distance of the vector $z = (x, y)$ from the origin. This is an absolute value. In particular, we have

$$|z_1 z_2| = |z_1||z_2|$$

for all z_1, z_2 in $\mathbb{C}$. Finally, we will denote ∂_x, ∂_y the partial derivatives with respect to x, y.

Let us now come to the central object of study.

Definition 1.1. Given a function

$$f : U \subset \mathbb{C} \to \mathbb{C}$$

where $U \subset \mathbb{C}$ is open, we say that f is *holomorphic* iff at every point z_0 in U the function f is *complex-differentiable*, i.e. if there is a point in $\mathbb{C}$, written $f'(z_0)$, such that

$$\frac{f(z_0+h)-f(z_0)}{h} \to f'(z_0)$$

whenever $h \to 0$ ($h \in \mathbb{C} - \{0\}$).

Obviously, *polynomials* over $\mathbb{C}$, i.e. functions of the form

$$p(z) = a_0 + a_1 z + ... + a_n z^n$$

with *coefficients* a_i in $\mathbb{C}$, are holomorphic over the whole complex plane $\mathbb{C}$. Note that complex differentiability is a quite strong notion of differentiability, since we are considering limits $h = h_1 + ih_2 \to 0$ (h_1, h_2 real) where we have $h_1 \to 0$ and $h_2 \to 0$ independently. Letting in particular $h = h_1 + i \cdot 0 \to 0$, we see that $\partial_x f(z_0) = f'(z_0)$ and letting $h = 0 + ih_2 \to 0$, we get $-i\partial_y f(z_0) = f'(z_0)$. From this it follows that

$$\partial_{\bar{z}} f(z_0) = 0 \tag{1.1}$$

where we let $\partial_{\bar{z}} := \partial_x + i\partial_y$. Writing $f = u + iv$ where u, v are functions $U \to \mathbb{R}$, we see that (1.1) is equivalent to the *Cauchy-Riemann-equations*

$$\partial_x u(z_0) = \partial_y v(z_0) \text{ and } \partial_x v(z_0) = -\partial_y u(z_0)$$

being fulfilled.

Let us now recall that a function $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at a point v_0 in U iff there is a real 2×2 matrix A and a norm $\|\cdot\|$ on $\mathbb{R}^2$ such that

$$\frac{\|F(v_0+h) - F(v_0) - Ah\|}{\|h\|} \to 0$$

as $h \to 0$ ($h \neq 0$). Since all norms in $\mathbb{R}^2$ are equivalent, the above notion of differentiability is independent of the choice of $\|\cdot\|$. Also, as one verifies, A is the Jacobian matrix given by $(\partial_j F_i(v_0))_{1 \leq i,j \leq 2}$.

Using our absolute value $|\cdot|$ on $\mathbb{C}$ defined above, which is of course a norm on $\mathbb{C}$ viewed , and considering again our holomorphic function $f : U \subset \mathbb{C} \to \mathbb{C}$, we have for every z_0 in U that

$$|(f(z_0+h) - f(z_0))/h - f'(z_0)| = |f(z_0+h) - f(z_0) - f'(z_0)h|/|h| \to 0$$

as $h \to 0$. So we see in particular that complex-differentiability implies differentiability. If we write down the matrix of our linear map $h \to f'(z_0)h = \partial_x f(x_0)h$ with respect to our (ordered) basis $1, i$ of $\mathbb{C}$, we get

$$\begin{pmatrix} \partial_x u(z_0) & -\partial_x v(z_0) \\ \partial_x v(z_0) & \partial_x u(z_0) \end{pmatrix}. \tag{1.2}$$

Lemma 1.2. Given a function $f : U \subset \mathbb{C} \to \mathbb{C}$, where U is an open subset of $\mathbb{C}$, then following are equivalent:

1. f is complex-differentiable.

2. f is differentiable and $\partial_{\bar{z}} f(z) = 0$ for all z in U.

Proof. We have already seen that 1. implies 2. Assume therefore that 2. holds and choose z_0 in U. Writing down the Jacobian of f with respect to our basis $1, i$ of $\mathbb{C}$, we get

$$\begin{pmatrix} \partial_x u(z_0) & \partial_y u(z_0) \\ \partial_x v(z_0) & \partial_y v(z_0) \end{pmatrix}.$$

Because of $\partial_{\bar{z}} f(z) = 0$, we may use the Cauchy-Riemann-equations on the second column of our matrix and get the matrix (1.2). From this 1. follows easily.

$\square$

If we have reviewed the facts about complex analysis we need, we will use them to look more closely at some properties of the Riemann zeta function.

2 Line integrals, Jordan curves and Cauchy's integral theorem I

The goal of this chapter is to prove a very famous and important theorem of Cauchy (with the help of Greens theorem).

We will first need some terminology. A *piecewise C^1 curve* in the complex plane $\mathbb{C}$ is any continuous map

$$\gamma : [a, b] \to \mathbb{C}$$

($a < b$) such that there is a decomposition of $[a, b]$

$$a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

for which the restriction of γ to each piece $[x_{i-1}, x_i]$ ($i = 1, 2, ..., n$) is one times continuously differentiable. We let $\mathrm{im}(\gamma) := \gamma([a, b])$ and inspired by real analysis define *the length of* γ, written $\mathrm{length}(\gamma)$, to be the integral

$$\int_a^b \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2}dt = \int_a^b |\gamma'(t)|dt\,,$$

where $\gamma = \gamma_1 + i\gamma_2$ with real-valued functions γ_1, γ_2.

Recall that a complex 1-form on an open subset U of $\mathbb{C}$ is an expression

$$\alpha(z) = f_1(z)dx + f_2(z)dy$$

where f_1, f_2 are maps $U \subset \mathbb{C} \to \mathbb{C}$. These forms obviously can be added to each other and multiplied by functions $f : U \subset \mathbb{C} \to \mathbb{C}$ and form a module of rank two over the ring of all complex valued functions defined on U.

Given such a form α, where we assume f_1, f_2 to be continuous and given a piecewise C^1 curve $\gamma = \gamma_1 + i\gamma_2 : [a, b] \to \mathbb{C}$ ($a < b$) such that $\gamma([a, b]) \subset U$, we define the *line integral of* α *with respect to* γ as

$$\int_\gamma \alpha := \int_a^b f_1(\gamma(t))\gamma_1'(t) + f_2(\gamma(t))\gamma_2'(t)dt.$$

We call any piecewise C^1 map $\psi : [c, d] \to [a, b]$ which is surjective and for which $\psi' > 0$ a *parameter transformation from* $[c, d]$ *to* $[a, b]$. Note that this notion of parameter transform respects the direction we travel on the curve. Now for γ as above also $\gamma \circ \psi$ will be a piecewise C^1curve and using the substitution rule of integration, we see that

$$\int_\gamma \alpha = \int_{\gamma\circ\psi} \alpha,$$

i.e. the integral is invariant of how we parametrize γ. Note that also the quantities $\mathrm{im}(\gamma)$ and $\mathrm{length}(\gamma)$ do not depend on how we parametrize γ. Given in particular a continuous function $f : U \subset \mathbb{C} \to \mathbb{C}$ and defining the 1-form

$$dz := dx + idy,$$

we may integrate the form

$$\alpha(z) = f(z)dz = f(z)dx + if(z)dy$$

and get

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Let us now come to an important example. For some $r > 0$, let

$$\gamma : [0, 2\pi] \to \mathbb{C}, t \mapsto z_0 + re^{it},$$

then $\gamma([0, 2\pi])$ is nothing but a circle of radius r around the point z_0. It is the boundary of the open disc $D = D(z_0, r)$ of radius r around z_0. Letting z be in D and $q := z - z_0/r$ and noting that $|q| < 1$ and $\gamma'(t) = ire^{it}$, we see that

$$\int_\gamma \frac{1}{w-z}dw = \int_0^{2\pi} \frac{ire^{it}}{z_0 - z + re^{it}}dt = i\int_0^{2\pi} \frac{1}{1 - qe^{-it}}dt.$$

Using the famous geometric series $1/(1-u) = \sum u^k$ for $|u| < 1$, we have

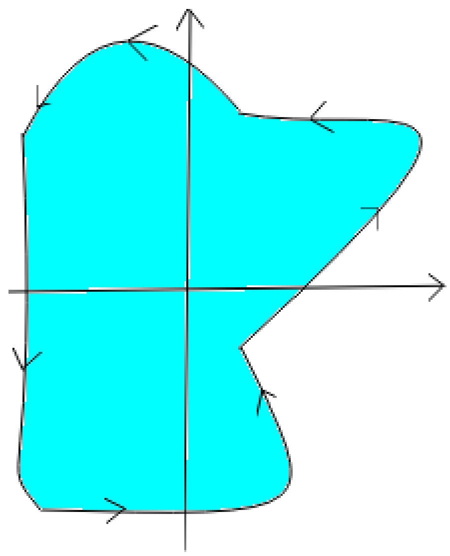
$$i\int_0^{2\pi} \sum q^k e^{-itk}dt = i\sum q^k \int_0^{2\pi} e^{-itk}dt = 2\pi i,$$

where we are allowed to interchange integration with summation (!). Putting everything together, we see that

$$\int_{\partial D} \frac{1}{w-z}dw = 2\pi i$$

for every $z \in D$. Note that our integral doesn't depend on the radius r we chose. Before we now get to the afore mentioned theorem of Cauchy, let us introduce more terminology:

A piecewise C^1 curve γ is said to be *of Jordan type* iff we have $\gamma(a) = \gamma(b)$, γ restricted to $[a, b[$ is injective and lastly $\mathbb{C} - \mathrm{im}(\gamma)$ consists of exactly two connected components of which exactly one is relatively compact, we call the relatively compact one *the interior* and denote $\mathrm{int}(\gamma)$. F.e. the curve we used in our example above, the circle arount a point z_0 of some radius r, is of Jordan type. Given a piecewise C^1 curve of Jordan type, we say that γ is *positively oriented*, iff "traveling" on γ, the interior should be on the left:



Finally, let us recall *Greens theorem*: Let $D \subset \mathbb{C}$ be the interior of a positively oriented piecewise C^1 curve γ of Jordan type, then given two C^1 functions $u, v : U \subset \mathbb{C} \to \mathbb{R}$ where U is open and contains the closure of D, then

$$\int_\gamma u dx + v dy = \int_D \partial_x v - \partial_y u dx dy.$$

Theorem 2.1. *(Cauchy's integral theorem) We let γ be a positively oriented piecewise C^1 curve of Jordan type. Then, if $f : U \subset \mathbb{C} \to \mathbb{C}$ is a holomorphic function which contains the closure of the interior of γ, then*

$$\int_\gamma f(z) dz = 0.$$

Proof. Let D be the interior of γ and write $f = u + iv$, where u, v are realvalued. Then by Greens theorem

$$\begin{aligned} \int_\gamma f(z) dz &= \int_\gamma u dx - v dy + i \int_\gamma v dx + u dy \\ &= \int_D -\partial_x v - \partial_y u dx dy + i \int_D \partial_x u - \partial_y v dx dy \\ &= i \int_D \partial_{\bar{z}} f(z) dx dy = 0 \end{aligned}$$

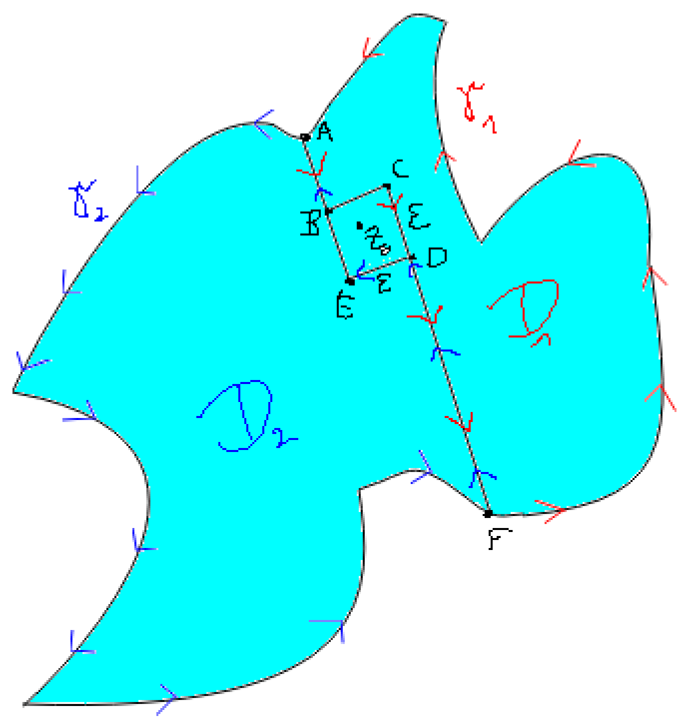
by the charactarization of holomorphic functions given in Lemma 2 of our previous chapter. $\square$

3 Cauchy's integral theorem II and Cauchy's integral formula

In this chapter, we extend Cauchys theorem from the last chapter a little bit and derive a corollary from it (another theorem of Cauchy).

In what follows, we let $f : U \subset \mathbb{C} \to \mathbb{C}$ with U open be a continuous function. Within U we assume lies a piecewise C^1 curve γ of Jordan type (in the picture below, this is represented as the boundary of the green colored region) together with its interior $\operatorname{int}(\gamma)$. We will further assume f to be holomorphic on U except for one point z_0 in the interior of γ.

Since z_0 is an interior point, we may draw a small (closed) square $BCDE$ (see the following picture) of side length $\epsilon > 0$ around z_0 which lies in the interior of γ. Extending the lines EB and CD to the boundary, we get two piecewise C^1 curves of Jordan type, namely $\gamma_1 = \gamma_{1,\epsilon}$, following the red arrows and traveling along A, B, C, D, F, A, and $\gamma_2 = \gamma_{2,\epsilon}$, following the blue arrows through A, F, D, E, B, A, with interiors D_1 and D_2:



Now we let $\gamma_0 = \gamma_{0,\epsilon}$ be the closed curve we get when we travel along B, C, D, E, B. With the help of Theorem 2.1 from the previous chapter applied to f and the curves $\gamma_{1,\epsilon}$ and $\gamma_{2,\epsilon}$ respectively, it follows that

$$0 = \int_{\gamma_{1,\epsilon}} f dz + \int_{\gamma_{2,\epsilon}} f dz = \int_{\gamma_{0,\epsilon}} f dz + \int_{\gamma} f dz,$$

where we use that in the first sum the integrals over the lines AB and DF cancel each other out since we integrate in different directions. So, since f is continuous and therefore bounded on the closure of $\mathrm{int}(\gamma)$ (which is compact), letting $c := \sup_{z \in \mathrm{int}(\gamma)} |f(z)| < \infty$ gives

$$\left| \int_{\gamma_{0,\epsilon}} f dz \right| \leq \mathrm{length}(\gamma_{0,\epsilon}) \cdot c = 4\epsilon c \to 0$$

as $\epsilon \searrow 0$. So we have the following extension of our theorem of Cauchy we presented in the last chapter (which is also due to Cauchy):

Theorem 3.1. *(Cauchy's integral theorem) We let γ be a positively oriented piecewise C^1 curve of Jordan type. We let further $f : U \subset \mathbb{C} \to \mathbb{C}$ be a continuous function where U is open and contains the closure of the interior int(γ) of γ. Then if f is holomorphic on U except for possibly one point z_0 in int(γ), we have*

$$\int_{\gamma} f(z) dz = 0.$$

Denoting $D(r, z_0) \subset \mathbb{C}$ the (open) disc of radius r and center z_0 and $\partial D(r, z_0)$ its positively oriented boundary, we may easily deduce:

Theorem 3.2. *(Cauchy's integral formula) Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Let z_0 be in U and $r > 0$ for which the closure of $D := D(r, z_0)$ lies in U, then we have for every z in D*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} dw.$$

Proof. Let z be in D and let

$$g(w) = \begin{cases} \frac{f(w) - f(z)}{w - z} & \text{for } w \neq z \\ f'(w) & \text{for } w = z \end{cases}.$$

Since f is holomorphic on U, g is both holomorphic on $U - \{z\}$ and continuous on $w = z$ and therefore on U. Applying g and ∂D to Theorem 3.1 above and recalling that in the previous chapter we found that $\int_{\partial D} 1/(w - z) dw = 2\pi i$, we get the assertion of our theorem:

$$\int_{\partial D} \frac{f(w)}{w - z} dw = \int_{\partial D} g(w) dw + \int_{\partial D} \frac{f(z)}{w - z} dw = 2\pi i f(z).$$

$\square$

Note that Theorem 2 is rather remarkable. By differentiating under the integral sign with respect to z (we leave it to the reader to check that we are allowed to do this) and therefore essentially differentiating the function $1/(w-z)$ with respect to z, we find that *a holomorphic function is in fact infinitely many often complex differentiable* (!):

Corollary 3.3. *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function, then f is infinitely often complex differentiable with*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{n+1}} dw,$$

where $D = D(r, z_0)$ is as in Theorem 2.

4 Parameter integrals, power series and analytic continuation

In this lecture, we first discuss two methods of how we can get new holomorphic functions out of old ones. The first method is via a parameter integral:

Proposition 4.1. *Suppose that we have a piecewise C^1 curve $\gamma : [a,b] \to \mathbb{C}$ and a continuous function $F : im(\gamma) \times U \to \mathbb{C}$ with $U \subset \mathbb{C}$ open. Suppose further that for every $w_0 \in im(\gamma)$ the function $z \mapsto F(w_0, z)$ on U is complex differentiable and that the function $(w,z) \mapsto \partial_z F(w,z)$ is continuous. Then the function*

$$G : U \to \mathbb{C}, z \mapsto \int_\gamma F(w,z)dw$$

is also complex differentiable with $G'(z) = \int_\gamma \partial_z F(w,z)dw$.

Proof. Let z be in U. Since F is continuous, we have that for every disc $D = D(0,r)$ with center zero for which $D + z \subset U$ is relatively compact, the function

$$\psi(w,h) := \begin{cases} F(w, z+h) - F(w,z)/h & \text{for } h \neq 0 \\ \partial_z F(w,z) & \text{for } h = 0 \end{cases}$$

is uniformly continuous on $\mathrm{im}(\gamma) \times D$. So integrating $\psi(w,h)$ along γ and letting $h \to 0$, we get the assertion of our proposition. $\square$

As a second method we now show that we may get holomorphic functions out of locally uniformly convergent series of holomorphic functions:

Proposition 4.2. *Suppose we have a sequence $(f_n)_{n\geq 0}$ of holomorphic functions $U \subset \mathbb{C} \to \mathbb{C}$. Then, if the infinite series $\sum_{n=0}^\infty f_n$ converges locally uniformly on U to the function $f : U \to \mathbb{C}$, then f itself is holomorphic and we have that $f^{(n)}$ is given by the infinite series $\sum_{k=0}^\infty f_k^{(n)}$, which is also locally uniformly convergent.*

Proof. First note that since $f = \sum f_k$ is locally uniformly convergent and the f_k are in particular continuous, f must be continuous (proof is left to the reader). Let $D \subset U$ be a relatively compact disc of radius > 0 with positively oriented boundary ∂D, then (f_k) converges on D uniformly. By definition this means that given $\epsilon > 0$, there is an integer N such that

$$|f(z) - \sum_{k\leq m} f_k(z)| < \epsilon$$

for all $m \geq N$ and $z \in D$. Choosing $n \geq 0$ and letting

$$F_n(z) := \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{n+1}} dw,$$

which is of course holomorphic on D by Proposition 4.1, then for every compact subset K of D and every z in K

$$\left| F_n(z) - \sum_{k \leq m} f^{(n)}(z) \right| \leq \frac{n!}{2\pi} \int_{\partial D} \left| \frac{f(w) - \sum_{k \leq m} f_k(w)}{(w-z)^{n+1}} \right| dw \leq C_n \cdot \epsilon,$$

where $C_n = n!\text{length}(\partial D)/2\pi \text{dist}(K, \partial D)^{n+1} > 0$ with

$$\text{dist}(K, \partial D) := \inf\{|z_1 - z_2| : z_1 \in K, z_2 \in \partial D\} > 0$$

and where we have used Cauchy's integral formula for the $n-$th derivative of a holomorphic function. Now, since D, ϵ and K were chosen arbirary, we see for all n that $F_n = \sum f_k^{(n)}$ locally uniformly, but since $F_0 = f$ by assumption, we have that f is holomorphic and that $F_n = f^{(n)}$ for all $n \geq 0$. □

We will now look at complex power series. Given a point z_0 in the complex plane, then a *complex power series in z around z_0* is an infinite series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the numbers a_n are in $\mathbb{C}$. Recall that for a sequence $(b_n)_{n \geq 0}$ of real numbers one defines the *limes superior*, written $\overline{\lim}_{n \geq 0} b_n$, to be

$$\lim_{n \to \infty} \sup_k \{b_k : k \geq n\}$$

which, being the limes of a monotonically decreasing sequence, is a unique element in $[-\infty, \infty]$. To f above we may associate a unique element R out of $[0, \infty]$ given by $R = 1/L$, where $L = \overline{\lim}_{n \geq 0} \sqrt[n]{|a_n|}$. R is called the *convergence radius* of f. Furthermore, the open disc $D = D(z_0, R)$ around z_0 of radius R will be called the *(open) disc of convergence* of f. The names we gave R and D are justified by the following well known result on power series:

Theorem 4.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a complex power series with convergence radius R. Let further $D = D(z_0, R)$ be the disc of convergence of f. Then f converges absolutely and locally uniformly on D. It may converge at points on the boundary ∂D of D and it diverges outside the closure $\overline{D}$ of D.*

Proof. (omitted) □

Let us give two more facts about complex power series which we will use below:

Proposition 4.4. *Let $f(z) = \sum a_n(z - z_0)^n$ be a complex power series. Then the following hold:*

1. *f restricted to its disc of convergence gives a holomorphic function.*
2. *Let $g(z) = \sum b_n(z - z_0)^n$ be a second complex power series, and assume that both f and g have a positive radius of convergence. Suppose further that for a sequence $(w_k)_{k \geq 0}$ of points in $\mathbb{C} - \{z_0\}$ which converges to z_0 we have $f(w_k) = g(w_k)$ for almost all $k \geq 0$, then $a_n = b_n$ for all $n \geq 0$.*

Proof. Assertion 1. follows directly from Proposition 4.2 where we let $f_n(z) = a_n(z - z_0)^n$. For assertion 2., note first that $\lim f(w_k) = f(z_0) = g(z_0) = \lim g(w_k)$, i.e. we have $a_0 = b_0$. Assuming now that $a_k = b_k$ for $k = 0, 1, 2, ..., n$ for some $n \geq 0$, then the function

$$F(z) := \frac{f(z) - g(z)}{(z - z_0)^{n+1}}$$

is locally continuous around z_0 and we have $F(w_k) = 0$ for almost all k. So we have $\lim F(w_k) = 0$, i.e. $a_{n+1} = b_{n+1}$. By induction 2. follows. □

Having introduced complex power series, we may now formulate another characterization of a function being holomorphic.

Theorem 4.5. *A function $f : U \subset \mathbb{C} \to \mathbb{C}$ is holomorphic iff at every point z_0 in U and for every $R > 0$ such that the closure of $D(z_0, R)$ lies in U, we have that f restricted to $D(z_0, R)$ is a complex power series around z_0 with a convergence radius larger than R.*

Proof. "⇒": Suppose f is holomorphic, then by Corollary 3.3 of the previous chapter, f is infinitely many times complex differentiable. So given z_0 in U, we may consider the (formal) power series

$$F(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Using the integral representation of $f^{(n)}(z_0)/n!$ from Corollary 6.3, then for every $D = D(z_0, R)$ whose closure lies in U, we have

$$\left|\frac{f^{(n)}(z_0)}{n!}\right| = \left|\frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{(z - z_0)^{n+1}}dz\right| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot C \cdot R^{-(n+1)} = C/R^n.$$

where $C = \sup_{z \in D} |f(z)|$. From this we get $\overline{\lim}_{n \geq 0} \sqrt[n]{\left| \frac{f^{(n)}(z_0)}{n!} \right|} \leq 1/R$, using that for a positive constant c we have $\sqrt[n]{c} \to 1$ as $n \to \infty$. Since we may make D a little larger by slightly increasing R and have that the closure still stays in U, we see that f restricted to D is a complex power series around z_0 with convergence radius greater than R.

"$\Leftarrow$": Since at every point z_0 the function f a complex power series with positive convergence radius, it is holomorphic there. So f itself is holomorphic. $\square$

Let us now end this chapter with a remarkable fact about holomorphic functions. We say that an open subset G of $\mathbb{C}$ is a *region* (*"Gebiet"* in german) iff it cannot be written as a union of two non-empty and disjoint open sets. Then the following holds:

Theorem 4.6. *(Analytic continuation) Given two holomorphic functions $f, g : G \subset \mathbb{C} \to \mathbb{C}$, where G is a region in $\mathbb{C}$. Suppose that $f(z) = g(z)$ on a sequence of distinct points with limit point in G. Then $f(z) = g(z)$ throughout G.*

Proof. Combining Theorem 4.5 and Proposition 4.4 (2.), we see that f and g agree on a small open disc $D \subset G$ of positive radius.
We let $U_1 := D$ and for $n \geq 1$ build the set U_{n+1} out of U_n as the union of all relatively compact open discs in G whose centers are in U_n. Then obviously for all $n \geq 1$ we have $U_n \subset U_{n+1} \subset G$, U_n is open and we have $f(z) = g(z)$ on U_n by using Theorem 4.5 and Proposition 4.4 (2.). We let V be the open subset $\cup_{n \geq 1} U_n$ of G and let $V' := G - V$. Suppose now there was a z_0 in V' which was a boundary point of V. Then we would have that very close to z_0 there was a point w_0 in one of the sets U_n for which a relatively compact disc in G with center w_0 existed which contained z_0 , i.e. then $z_0 \in U_{n+1} \subset V$, a contradiction. So V' contains no boundary point of V and must therefore be open since G is open. But since $G = V \cup V'$ is a region and $V \neq \emptyset$, we must have $V' = \emptyset$, which concludes our proof. $\square$

5 Zeros and poles

Let us now talk about zeros and poles. Suppose we have a holomorphic function $f : U \subset \mathbb{C} \to \mathbb{C}$. A *zero* of f is any point z_0 in U such that $f(z_0) = 0$. By the analytic continuation theorem from the last chapter, it follows that on any region $G \subset U$ we must have that f is either completely zero or it has at most *isolated zeros,* i.e. the zeros of f have no limit point in G. Note that the latter means that for every isolated zero z_0 in U, we have $f(z_0) = 0$ and $f(z) \neq 0$ for all $z \neq z_0$ in a small non-empty open disc with center z_0. Given an isolated zero z_0 of f in U, then by writing f around z_0 as a complex power series (see Theorem 5 of our last chapter)

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

we see that there is a smallest integer $n \geq 1$ such that $a_n \neq 0$ and $a_k = 0$ for $k < n$. Note that by the uniqueness of the coefficients a_k (see Proposition 4.4 (2.) of the last chapter), we have that n is also unique. The number n is called the *order* (or the *multiplicity*) of our zero z_0. If $n = 1$ then z_0 is called a *simple zero.* So we have

$$f(z) = (z - z_0)^n [a_n + a_{n+1}(z - z_0) + ...] = (z - z_0)^n g(z),$$

where g is a holomorphic function which is non-vanishing in a small neighbourhood of z_0 and for which $g(z_0) = a_n \neq 0$.

To define what a pole is, let us recall that the *punctured disc* of radius R and center z_0, written $D'(z_0, R)$, is the open set

$$D(z_0, R) - \{z_0\} = \{z : 0 < |z - z_0| < R\}.$$

Suppose that the function f is holomorphic on a punctured disc $D' = D'(z_0, R)$ of positive radius R with center z_0. By the analytic continuation theorem, we may assume that f does not vanish on D'. We say that f has a *pole* at z_0 iff the function $1/f$ defined on D' can be extended holomorphically to a function of the full disc $D = D(z_0, R)$ with value 0 at z_0. By what we have reasoned above in the case of zeros, we have that there is a unique positive integer n with

$$1/f(z) = (z - z_0)^n g(z) \text{ or } f(z) = (z - z_0)^{-n} h(z),$$

where $h = 1/g$ is holomorphic and non-vanishing on the full disc $D = D(z_0, R)$. We call n the *order* of the pole at z_0 and call the pole a *simple pole* iff $n = 1$. Expanding h around z_0 as a complex power series (of positive convergence radius), renaming its unique coefficients appropriately and dividing the summands by $(z - z_0)^n$, we see that for every z in D' we have

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + ... + \frac{a_{-1}}{(z - z_0)} + r(z),$$

where $a_{-n} \neq 0$, the coefficients a_k are unique and where $r(z)$ is holomorphic on D. The sum above to the left of $r(z)$ is called the *principal part* of f at z_0 and a_{-1} is called the *residue* of f at z_0, written $\text{Res}_{z_0} f$.

Now, let C be a positively oriented circle in D, where z_0 is in the interior of C. Recall that by Cauchys integral formula from the last chapter, we have for every constant function $F(z) = a$

$$F^{(k)}(z_0) = \frac{k!}{2\pi i} \int_C \frac{a}{(z - z_0)^{k+1}} dz,$$

where $F^{(0)}(z_0) = a$ and $F^{(k)}(z_0) = 0$ for $k \geq 1$. Using this on the principal part of f at z_0 and using Cauchys integral theorem on r, we see that

$$\int_C f(z)dz = \sum_{k=1}^{n} \int_C \frac{a_{-k}}{(z - z_0)^k} dz + \int_C r(z)dz = 2\pi i a_{-1} = 2\pi i \text{Res}_{z_0} f. \quad (5.1)$$

Using what we have observed above, we can prove the famous *residue theorem*:

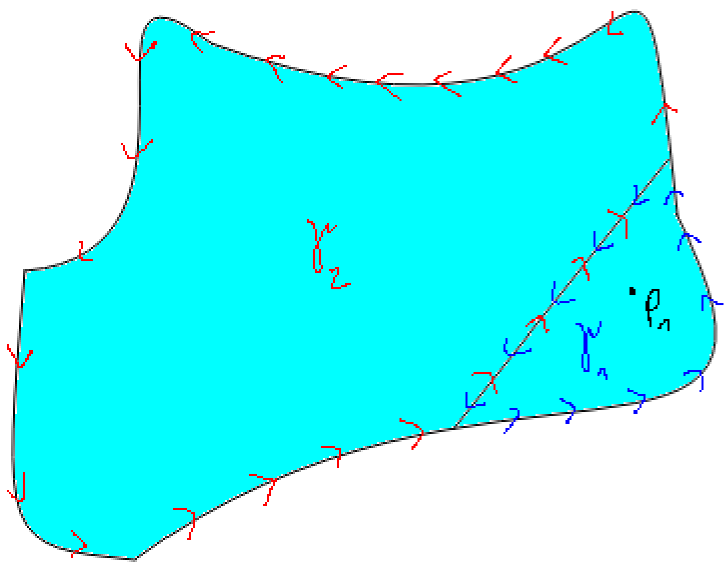
Theorem 5.1. *(Residue theorem) Let γ be a piecewise C^1 curve of Jordan type in the complex plane. Suppose further that we have a function f which is holomorphic on an open set containing γ and its interior, except for some poles at points $z_1, z_2, ..., z_n$ in the interior of γ. Then*

$$\int_\gamma f(z) = 2\pi i \sum_{k=1}^{n} Res_{z_i} f.$$

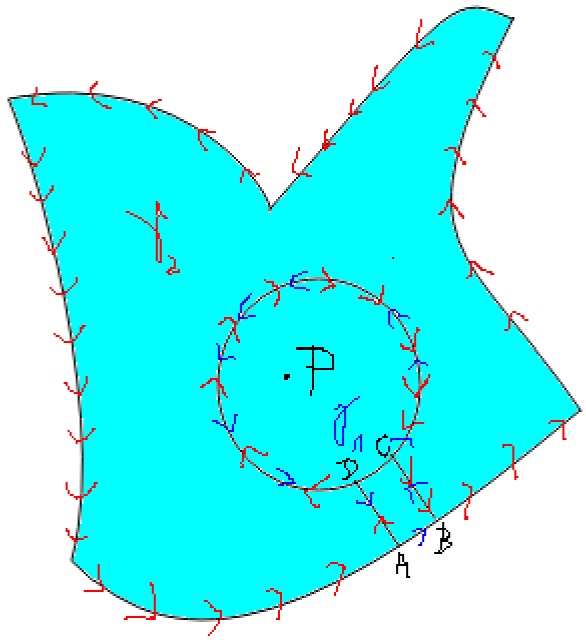
Proof. Writing

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$$

where γ_1 and γ_2 are the piecewise C^1 curves of Jordan type drawn in the following picture (for γ_1 we follow the blue arrows and for γ_2 we follow the red arrows, the boundary of the green area shows γ and during integration the additional path we introduced and which separates the interiors of γ_1 and γ_2 cancles out)



and where the interior of γ_1 contains exactly one pole P_1 of f, we see that we are by induction reduced to the case $n = 1$. For the proof of the case $n = 1$ we consider the following picture:



The circle around the pole P, which we will denote c, is assumed to be small enough such that the integral formula (5.1) holds. Now as in the picture indicated, we introduce, appart from γ which as always is depicted as the positively oriented boundary of the green region, two additional piecewise C^1 curves of Joran type, namely γ_1 and γ_2, where for γ_1 we travel along the blue arrows beginning say at D and going through A, B, C, around the circle to D again and where for γ_2 we travel along the red arrows beginning say at D going along the circle to C, then to B and along γ to A and to D again. Using Cauchys integral theorem on γ_2, we certainly have

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Introducing the piecewise C^1 curve of Jordan type γ_3 where we start say at C travel along c from C to D, then to A, to B and to C again, we see again by Cauchys integral theorem and the fact that during integration we have to pass the line between the points D, C shared by γ_3 and c in opposite directions,

$$\int_c f(z)dz = \int_c f(z)dz + \int_{\gamma_3} f(z) = \int_{\gamma_1} f(z)dz.$$

So putting everything together and using formula (5.1), we see that

$$\int_\gamma f(z)dz = \int_c f(z)dz = 2\pi i \mathrm{Res}_P f.$$

This concludes our proof. $\square$

6 Logarithms

In this chapter we we will talk about logarithms. To do that, let first us recall/reintroduce the exponential function. The *exponential function* is defined as the complex power series

$$\exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + z^2/2 + z^3/6 + \ldots .$$

Since $R = 1/\lim\sup_{k \geq 0} \sqrt[k]{1/k!} = \infty$, exp converges everywhere absolutely and locally uniformly. It therefore gives an example of a so called *analytic function*, i.e. a function which is holomorphic on the whole complex plane. Using the famous binomial theorem, we get

$$\frac{(z+w)^n}{n!} = \frac{1}{n!} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) z^k w^{n-k} = \sum_{k+l=n} \frac{z^k}{k!} \frac{w^l}{l!}.$$

Summing this over all $n \geq 0$, we see that for all complex numbers z, w we have

$$\exp(z+w) = \exp(z)\exp(w).$$

This is called the *functional equation* for the exponential function. Since in particular

$$1 = \exp(z)\exp(-z),$$

we see that exp vanishes nowhere on $\mathbb{C}$ and that it is positive on $\mathbb{R}$, since it is positive for every non-negative number and therefore has to be positive on the negative numbers. Also, since for x in $\mathbb{R}$ we have

$$\overline{\exp(ix)} = \exp(\overline{ix}) = \exp(-ix),$$

it follows also that $|\exp(ix)| = 1$, i.e. that $\exp(ix)$ lies on the unit circle. More precisely, introducing the complex power series

$$\cos(z) := \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \text{ and } \sin(z) := \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!},$$

which are, as one easily checks, both analytic, then obviously

$$\exp(iz) = \cos(z) + i\sin(z).$$

So we have in particular for real numbers x that

$$\exp(ix) = \cos(x) + i\sin(x) \text{ and } 1 = \cos(x)^2 + \sin(x)^2.$$

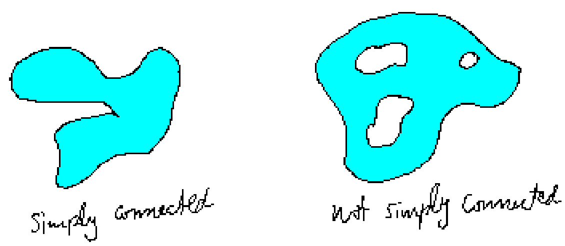
Finally, the reader may verify that using $\exp(ix)$, we hit every point on the unit circle. So every point in the complex plane, except zero, can be expressed as

$$\exp(x+iy) = r \cdot (\cos(\alpha) + i\sin(\alpha))$$

where we let $y = \alpha$ and $r = \exp(x) > 0$.

Having considered (briefly) the exponential function as a function on the complex plane, let us now come to logarithm functions. First some notation:

Given two points z_1, z_2 in the complex plane, then an *integration path from* z_1 *to* z_2 is any piecewise C^1 curve $\gamma : [a, b] \to \mathbb{C}$ $(a < b)$ which is injective on $[a, b[$ and for which $\gamma_z(a) = z_1, \gamma_z(b) = z_2$. The point z_1 is then called the *start point* and z_2 is called the *end point* of γ. As defined earlier, γ is called *closed* iff start and end point are equal and it is called *of Jordan type* iff it is closed and positively oriented. Let U be an open subset of $\mathbb{C}$. We call U *path-connected* iff for every pair z_1, z_2 of points in U there is an integration path from z_1 to z_2. Furthermore, we call U *simply connected*, iff U is path-connected and contains the interiors of all the closed integration paths within U:



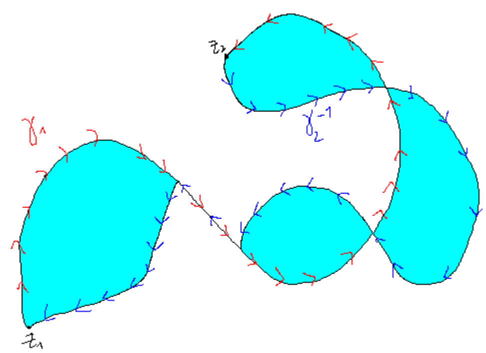
Finally, we call a subset U of $\mathbb{C}$ a *simply connected domain* or a *domain without holes* iff U is open and simply connected.

Using Cauchy's integral theorem, we first prove the following proposition:

Proposition 6.1. *Let γ_1, γ_2 be two integration paths which have the same start and end points and which lie within a simply connected domain $U \subset \mathbb{C}$. Then we have for every holomorphic function $f : U \subset \mathbb{C} \to \mathbb{C}$ that*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof. Let us denote the start point of our curves z_1 and their end point z_2. Denoting $\gamma_1\gamma_2^{-1}$ the closed curve where we travel first along γ_1 from z_1 to z_2 and then in reversed direction along γ_2 from z_2 back to z_1 (γ_2 reversed is usually denoted γ_2^{-1}), we get a closed curve which consists of finitely many closed integration paths that are connected by lines which are travelled in both directions exactly once:



Obviously, we have

$$\int_{\gamma_1 \gamma_2^{-1}} f(z)dz = \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz,$$

where we use the fact that integration over a curve in the opposite direction changes the sign of the integral. Since the integral on the left vanishes on closed integration paths by Cauchy's integral theorem and since the integration of a line in both directions exactly once gives zero, we have that our integral on the left is zero and the proposition follows. □

In light of the previous proposition, we may make the following definition:

Definition 6.2. Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ a holomorphic function. Then for every integration path γ from z_1 to z_2 in U, we write

$$\int_{z_1}^{z_2} f(z)dz := \int_\gamma f(z)dz.$$

As a consequence of our Proposition 6.1, we have the following corollary:

Corollary 6.3. *Let $U \subset \mathbb{C}$ be a simply connected domain and let z_0 be a point in U. Then for every holomorphic function $f : U \to \mathbb{C}$, the function*

$$F : U \to \mathbb{C}, z \mapsto \int_{z_0}^{z} f(w)dw$$

is holomorphic with $F' = f$. If $G' = f$ for another holomorphic function G on U, then $F = G + c$ for some constant c in $\mathbb{C}$.

Proof. Let z be in U. Then for all h with $z+h$ in U, we have by our proposition above

$$(F(z+h) - F(z))/h = \frac{1}{h} \int_{z}^{z+h} f(w)dw.$$

For h small enough, the line sement $\gamma(t) = z + th$ with $t \in [0,1]$ and where $\gamma'(t) = h$ lies within U and it follows

$$\frac{1}{h}\int_z^{z+h} f(w)dw = \frac{1}{h}\int_\gamma f(w)dw = \int_0^1 f(z+th)dt \to f(z)$$

as $h \to 0$. So the the first assertion of our collary follows. For the second assertion, note that $(F-G)' = 0$ on U and that therefore $F-G$, written locally as a powerseries, must be constant because of the uniqueness of powerseries coefficients. ☐

Definition 6.4. A *logarithm* of a holormorhic function $F : U \to \mathbb{C}$ is any holomorphic function $f : U \to \mathbb{C}$ such that

$$F = e^f$$

on U.

Theorem 6.5. *If $U \subset \mathbb{C}$ is a simply connected domain, then any holomorphic function $F : U \to \mathbb{C}$ which is non-vanishing on U possesses a logarithm. In fact, every logarithm of F can be written as*

$$f(z) = \int_{z_0}^{z} \frac{F'(w)}{F(w)}dw + C$$

for some fixed z_0 in U and some constant C in $\mathbb{C}$. In particular, two logarithms of F differ only by a constant.

Proof. First note that if f is a logarithm of F, then $F' = f'e^f = f'F$, or $f' = F'/F$, so by Corollary 6.3 above, f has an integral-representation as claimed in our theorem.

Let now $G(z) := e^{g(z)}$ where

$$g(z) = \int_{z_0}^{z} \frac{F'(w)}{F(w)}dw$$

with $z \in U$. We show that $cG = F$ for some non-zero constant c in $\mathbb{C}$. Then for every C with $c = e^C$, the function $f = g + C$ will be a logarithm of $F(z)$. We have

$$G'(z) = g'(z)G(z) = \frac{F'(z)}{F(z)}G(z)$$

or

$$G'(z)F(z) = F'(z)G(z) \tag{6.1}$$

on U. Fixing a w in U and expanding F, G around w as power series

$$F(z) = \sum_{k=0}^{\infty} a_k(z-w)^k \text{ and } G(z) = \sum_{k=0}^{\infty} b_k(z-w)^k,$$

where $a_0, b_0 \neq 0$ because F, G do not vanish everywhere on U, we will show by induction that $a_n = cb_n$ where $c = a_0/b_0$ and $n \geq 0$:

Obviously, we have $a_0 = cb_0$. Suppose now that for some $n \geq 0$, we have $a_k = cb_k$ when $n \geq k \geq 0$. Using equation (6.1) and comparing the $n-$th coefficient of both sides, we have

$$\sum_{k=0}^{n}(k+1)b_{k+1}a_{n-k} = \sum_{k=0}^{n}(k+1)a_{k+1}b_{n-k}$$

or

$$(n+1)b_{n+1}a_0 = (n+1)a_{n+1}b_0 + \sum_{k=0}^{n-1}(k+1)(a_{k+1}b_{n-k} - b_{k+1}a_{n-k}).$$

So since for $k = 0, .., n-1$ we have by induction

$$a_{k+1}b_{n-k} - b_{k+1}a_{n-k} = cb_{k+1}b_{n-k} - cb_{k+1}b_{n-k} = 0,$$

it follows $a_{n+1} = cb_{n+1}$. So since U is in particular a region, we have $F = cG$ by analytic continuation. This concludes our proof. $\square$

7 Meromorphic functions and products

In this lecture, we look at some well known facts about products. Given an open subset U of $\mathbb{C}$, then a *meromorphic function* on U is any holomorphic function $f : U - N_f \to \mathbb{C}$, where N_f is a discrete subset of U where f has poles. F.e., given a non-zero polynomial $p(z)$, then $f(z) = 1/p(z)$ is a meromorphic function on $\mathbb{C}$ where N_f is the set of zeros of $p(z)$. N_f in this case is not only discrete but also finite. In fact Gauss proved, as we will do also shortly using Complex analysis, that $p(z)$ is a product of (finitely many) linear factors:

Theorem 7.1. *(Fundamental Theorem of Algebra) Every complex polynomial of positive degree has a zero.*

Since for every non-zero polynomial of $p(z)$ positive degree we have $p(z) \to \infty$ as $z \to \infty$, the Theorem 7.1 follows from

Theorem 7.2. *(Liouville) Given a holomorphic function f on the complex plane which has no zeros and such that $1/f$ is bounded, then f must be constant.*

Proof. By assumption, there is a real number $c > 0$ such that $|1/f(z)| \leq c$ for all complex numbers z. Given a complex number z_0, then we have for every circle C_r of radius r and center z_0 that

$$|(1/f)'(z_0)| = \left| \frac{1}{2\pi i} \int_{C_r} \frac{1}{f(z)(z-z_0)^2} dz \right| \leq \frac{1}{2\pi} 2\pi r c r^{-2} = \frac{c}{r}$$

for all $r > 0$. Letting $r \to \infty$, we see that $(1/f)'(z_0) = 0$. And since c was arbitrary, we see that $(1/f)'(z) = 0$ for all z in the plane, i.e. $1/f$ and therefore also f must be constant. $\square$

Obviously, given two meromorphic functions f_1, f_2 on an open subset U of the complex plane, then also their sum $f_1 + f_2$ and their (pointweise) product $f_1 f_2$ is meromorphic. Moreover, if f_1, f_2 are meromorphic with f_2 being non-trivial, i.e. not completely zero, in any region within U, then f_1/f_2 is meromorphic since by analytic continuation f_2 can only have a discrete set of zeros.

A *divisor* D on the complex plane will be any function $D : \mathbb{C} \to \mathbb{Z}$ where the support of D is a discrete subset of $\mathbb{C}$. It is sometimes convenient to write

$$D = \{(a_0, n_0), (a_1, n_1), ...\}$$

where $\{a_0, a_1, ...\}$ is a discrete set of points in $\mathbb{C}$ and where the numbers $n_0, n_1, ...$ are non-zero integers. Obviously, given a nowhere trivial meromorphic function $f(z)$ on the complex plane, we may form the divisor D_f where $D_f(a) = n$ when a is a zero of order n and $D_f(a) = -n$ if a is a pole of order n.

Question: Given an arbitrary divisor D on $\mathbb{C}$, is there a meromorphic function f with $D = D_f$?

As we will see below, the well-known answer to this question is *yes*! Since obviously

$$D_{fg} = D_f + D_g \text{ and especially } D_{1/f} = -D_f,$$

we are reduced to the case where $D(z) \geq 0$ for all z, i.e. the case where D is *positive*. Also, if supp(D) is a finite set, we have $D = D_{p/q}$ where $p(z)$ and $q(z)$ are appropriately chosen polynomials. So in sum we need to deal with the case where supp(D) is infinite (and therefore unbounded) and where D is positive.

In what follows we let $\log(z)$ be the *natural branch of the logarithm*, i.e. we let

$$\log(z) = \int_1^z \frac{1}{w} dw$$

where z is in $\mathbb{C} - \mathbb{R}_{\geq 0}$. Before we proceed let us list some properties of $\log(z)$:

Lemma 7.3. *The following are true:*

1. *We have the power series expansion around zero*

$$-\log(1-z) = \sum_{n \geq 1} \frac{z^n}{n}$$

with radius of convergence $R = 1$.

2. *For $|z| \leq 1/4$ we have*

$$\frac{2}{3}|u| \leq |\log(1+u)| \leq \frac{4}{3}|u|.$$

3. *Let $e_N(z) = \log(1-z) + \sum_{n=1}^{N} \frac{z^n}{n}$. Then for every $N \geq 0$ and every $|z| \leq 1/4$ we have*

$$|e_N(z)| \leq 4/3 \, |z|^{N+1}.$$

4. *Let $E_n(z) = \exp(e_N(z)) = (1-z)\exp(\sum_{k=1}^{n} \frac{z^k}{k})$. Then whenever $n \geq 0$ and $|z| \leq 1/4$, we have*

$$|1 - E_n(z)| \leq 2|z|^{n+1}.$$

Proof. To prove 1., note that the power series on the right clearly is absolutely and locally uniformly convergent on the open unit disc and it diverges outside its closure, so it has a convergence radius of one. Furthermore, within a small disc around zero, both sides have the same derivative, namely

$$\frac{1}{1-z} = \sum z^n.$$

In particular up to the constant term, both sides have the same power series expansion around zero. But since $\log(1-0) = 0$, they have the same.

To prove 2., we use 1.: Since for $|z| \leq 1/4$

$$\sum_{n \geq 2} \frac{|z|^n}{n} \leq \sum_{n \geq 0} |z|^{n+2} \leq 1/4 \cdot |z| \cdot \frac{1}{1-|z|} \leq 1/3|z|,$$

we have

$$2/3|z| = |z| - 1/3|z| \leq |\log(1-z)| \leq |z| + 1/3|z| = 4/3|z|.$$

Again using 1., we prove 3.:
We have

$$\left|\log(1-z) + \sum_{n=1}^{N} \frac{z^n}{n}\right| = \left|\sum_{n\geq N+1} \frac{z^n}{n}\right| \leq |z|^{N+1}\frac{1}{1-|z|} \leq 4/3|z|^{N+1}.$$

Finally, 4. is simply a consequence of 3.: We have

$$|1 - E_n(z)| \leq \sum_{k\geq 1} \frac{|e_n(z)|^k}{k!} \leq \frac{4}{3}|z|^{n+1} \sum_{k\geq 0}\left(\frac{4}{3}|z|^{n+1}\right)^k = \frac{\frac{4}{3}|z|^{n+1}}{1-\frac{4}{3}|z|^{n+1}} \leq 2|z|^{n+1}.$$

□

Inspired by Lemma 7.3 (2.) and the continuity of the exponential function, we define:

Definition 7.4. We say that the infinite product $\prod_{n\geq 0}(1-a_n)$ *converges absolutely* iff $\sum \log(1-a_n)$ converges absolutely. Moreover, we say that a product $\prod(1-f_n(z))$ converges *absolutely and locally uniformly* on a subset U of the complex plane iff $\sum \log(1-f_n(z))$ does.

Clearly, in the case of convergence, the product $\prod(1-f_n(z))$ will be a holomorphic function whenever the functions $f_n(z)$ are. Let us now come to the main theorem of this chapter:

Theorem 7.5. *(Weierstrass) Given a divisor*

$$D = \{(a_0, n_0), (a_1, n_1), (a_2, n_2), ...\},$$

where $0 = |a_0| < |a_1| \leq |a_2| \leq |a_3| \leq ...$ *and* $a_n \to \infty$ *as* $n \to \infty$*, then the function*

$$f(z) := z^{n_0} \prod_{k\geq 1}(E_{k+n_k}(z/a_k))^{n_k}$$

defines a holomorphic function on the complex plane and satisfies $D = D_f$*. If* $g(z)$ *is another such function then* $f/g = e^h$ *for a holomorphic function* h *on* $\mathbb{C}$*.*

Proof. Let $R > 0$. Since $a_k \to \infty$ as $k \to \infty$, there is a constant $k_0 \geq 0$ such that $|z/a_k| \leq 1/4$ for all $k \geq k_0$ and all z with $|z| \leq R$. So using Lemma 7.3 (4.), we have

$$\begin{array}{rcl}\sum_{k\geq k_0} n_k\,|E_{k+n_k}(z/a_k) - 1| & \leq & \sum_{k\geq k_0} 2n_k\,|z/a_k|^{k+n_k+1} \\ & \leq & \sum_{k\geq k_0} \frac{n_k}{4^{n_k}} \cdot \frac{1}{4^{k+1}} \\ & \leq & \frac{1}{1-1/4} = 4/3 < \infty.\end{array}$$

This means that $f(z)$ converges absolutely and uniformly on the closed disc around zero with radius R, and since R was arbitrary $f(z)$ forms an absolutely and locally uniformly convergent product of holomorphic functions and is therefore itself holomorphic. Obviously, by construction, we have $D = D_f$. Finally, from what we know about logarithms, if the holomorphic function $g(z)$ on the complex plane also satisfies $D = D_g$ then the meromorphic function f/g defines a function which has no poles and no zeros on $\mathbb{C}$ and therefore has a logarithm on $\mathbb{C}$. This proves the last assertion of the theorem. $\square$

8 The gamma function and the Riemann zeta function

In this chapter, we get a first glance at the famous Riemann zeta function and thereby give examples for the topics we have developed so far. Our introduction to this function is absolutely standard and follows Riemanns very beautiful article [1], where beautiful refers to both its very influential and important content and also in how it is written; this article is de facto a master piece, in it, Riemann formulates what is today known as the *Riemann hypothesis.*

Given a complex number $s = \sigma + it$ with $\sigma > 1$, we introduce the *Riemann zeta function* ζ as the infinite series

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}.$$

Since $|n^{-s}| = |n^{-\sigma}| \leq \int_{n-1}^{n} x^{-\sigma} dx$ for $n > 1$, we see that $|\zeta(s)| \leq 1 + \int_{x>1} x^{-\sigma} dx = \sigma/(\sigma - 1)$ and therefore that $\zeta(s)$ converges absolutely and locally uniformly when $\sigma > 1$ and consequently forms a holomorphic function by Proposition 4.2. Using the uniqueness of the factorization of natural numbers into prime powers, we immediately get the important fact that for $\sigma > 1$ we have

$$\zeta(s) = \sum n^{-s} = \prod_{p} \sum_{k\geq 0} p^{-ks} = \prod_{p} (1 - p^{-s})^{-1}$$

where p runs through the set of prime numbers (this f.e. has been used by Euler in the case where s is real).

To learn more about $\zeta(s)$ as a complex function, Riemann used the *Gamma function* which is defined as the integral

$$\Gamma(s) := \int_{u>0} e^{-u} u^{s-1} du.$$

This integral converges absolutely for every $\sigma > 0$ as a simple calculation shows. Then $\Gamma(s) = \sum_{n\geq 1}(\Gamma_{n+1}(s) - \Gamma_n(s))$ where $\Gamma_m(s) := \int_{1/m}^{m} e^{-u} u^{s-1} du$ converges absolutely and uniformly. Using Proposition 4.1 and Proposition 4.2, we see that $\Gamma(s)$ forms a holomorphic function for all s where $\sigma > 0$.

Now, using integration by parts, we see that

$$\Gamma(s) = e^{-u} u^s/s|_0^\infty + \int_{u>0} e^{-u} u^s/s du = \frac{1}{s}\Gamma(s + 1)$$

for $\sigma > -1$ forms a meromorphic function with a simple pole at $s = 0$, since $\Gamma(1) = \int_{u>0} e^{-u} du = 1 \neq 0$. Repeatedly applying this technique, we see that we may extend $\Gamma(s)$ meromorphically to the whole complex plane with simple poles at $s = 0, -1, -2, -3, ...$ and satisfying the functional equation

$$\Gamma(s + 1) = s\Gamma(s).$$

Now let us see how cleverly Riemann shows that ζ also can be extended meromorphically to the complex plane and that it also satisfies a functional equation. We let

$$\Theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = 1 + 2\omega(x)$$

where

$$\omega(x) = \sum_{n\geq 1} e^{-\pi n^2 x}.$$

Both Θ and ω converge uniformly for $x \geq \epsilon > 0$. Moreover, one can check (may be we will do that in a later chapter using complex analysis) that Θ which is an example of a so called "theta function" satisfies the functional equation

$$\Theta(1/x) = \sqrt{x}\Theta(x)$$

or equivalently

$$\omega(1/x) = -1/2 + \sqrt{x}/2 + \sqrt{x}\omega(x).$$

Now using the uniform convergence of ω, we see that

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-s/2}\sum_{n\geq 1} n^{-s}\int_{u>0} e^{-u}u^{s/2-1}du = \int_{v>0}\omega(v)v^{s/2-1}dv,$$

where we get the last equality by letting $u = \pi n^2 v$. The right hand side of the last equation is equal to

$$\int_0^1 \omega(v)v^{s/2-1}dv + \int_1^\infty \omega(v)v^{s/2-1}dv$$

which after letting $v \mapsto 1/v$ in the first integral and applying the functional equation of ω and summarizing terms results in the identity

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = -1/s + 1/(s-1) + \int_1^\infty \omega(v)(v^{s/2-1} + v^{-s/2-1/2})dv. \quad (8.1)$$

The identity (8.1) is remarkable: The integral on the right hand side converges for all s locally uniformly (note that $\omega(x) = O(e^{-\pi x})$) and therefore forms a holomorphic function on the complex plane. Since Γ has a (simple) pole at zero, we see therefore that ζ is in fact a meromorphic function on the whole complex plane with only one simple pole at $s = 1$. But not only that, the right hand side of (8.1) is invariant under the transformation $s \mapsto 1 - s$, so we get a functional equation. Now consider the zeros of ζ for $\sigma < 0$: ζ must have zeros of order one at $s = -2, -4, -6, \ldots$ since the right hand side of (8.1) is holomorphic for $\sigma < 0$, obviously positive (i.e. non-zero) for real $s < -1$ and Γ has simple poles at $s = -1, -2, \ldots$. Any other zero at $\sigma < 0$ of ζ would lead by the functional equation to a zero of Γ at $\sigma > 1/2$ or ζ at $\sigma > 1$ where as one checks they have no zeros. So we have the following result:

Theorem 8.1. *(Riemann) $\zeta(s)$ can be meromorphically extended to the whole complex plane. More precisely, in $\mathbb{C} - \{1\}$, $\zeta(s)$ is holomorphic and it has a simple pole at $s = 1$. Moreover, we have*

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s).$$

For $\sigma < 0$, $\zeta(s)$ has precisely the zeros $s = -2, -4, -6, \ldots.$ and they are of order one.

Remark 8.2. There is of course a lot more to say, f.e. the zeros mentioned in the theorem have all order one. And there is the critical strip we have not mentioned yet. This will come hopefully later ...

[1] Riemann, Bernhard, Ueber die Anzahl der Primzahlen unter einer gegebenen Groesse, Monatsberichte der Koeniglichen Preussischen Akademie der Wissenschaften zu Berlin. Aus dem Jahre 1859. S. 671–680.

https://www.claymath.org/sites/default/files/zeta.pdf